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Stability Analysis of Higher-Order Delta-Sigma Modulators using the Describing Function Method

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Abstract- **The aim of this paper is to determine the stability of higher-order Δ-Σ modulators using the Describing Function Method. The maximum stable input limits for third-, fourth- and fifth-order Chebyshev Type II based Δ-Σ modulators are established. These results are useful for optimising the design of higher-order Δ-Σ modulators.**

I. INTRODUCTION

The stable input amplitude limits for Δ-Σ modulators is complicated to predict due to the non-linearity introduced by the quantizer in the feedback loop. Various approaches have been employed to explain this nonlinear behaviour. Using quasilinear modeling, a new interpretation of the instability mechanism for Δ-Σ modulators based on the noise amplification curve is given in [1]. This is restricted for dc inputs and unity quantizer gains. The quasilinear method can be extended to more than one input with each input represented by a separate equivalent gain. This concept forms the basis for the Describing Function (DF) method [2]. In this paper, the stability analysis based on the noise amplification curve is accomplished using the DF method for dc and sinusoidal inputs for non-unity quantizer gain values. In Section II, the quasilinear stability of Δ-Σ modulators is explained based on the noise amplification curve. In Section III, the derivation of the noise amplification curves for dc and sinusoidal inputs with the DF method is shown. The simulation results are illustrated and discussed in Section IV concluding with Section V.

II. QUASILINEAR STABILITY ANALYSIS OF Δ-Σ MODULATORS

A generic Δ-Σ modulator having its quantizer replaced by a gain factor K followed by additive quantization noise $q(k)$ [1] is shown in Figure 1.

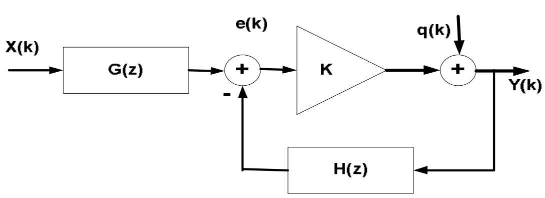

Figure 1. Quasilinear Δ-Σ modulator Quantizer Model.

The output of the modulator in the z-domain is given by

$$Y(z) = STF(z)X(z) + NTF(z)Q(z) \quad (1)$$

where, $Y(z)$, $X(z)$ and $Q(z)$ are the z-transforms of the output, input and quantizer noise signals respectively. Also, $STF(z)$ and $NTF(z)$ are the Signal and Noise Transfer functions of the Δ-Σ modulator derived from Figure 1.

$$STF(z) = \frac{K.G(z)}{1 + K.H(z)} \quad (2)$$

$$NTF(z) = \frac{1}{1 + K.H(z)} \quad (3)$$

Equations (2) and (3) show that the poles of the denominator $(1+KH(z))$ determine the stability of the modulator. For a given $H(z)$, there will be a certain interval $[K_{min}, K_{max}]$ for which the modulator is stable [3]. Assuming $q(k)$ to be Gaussian white stochastic $G(0, \sigma_q^2)$ and the transfer function between $q(k)$ and $y(k)$ to be known, then the output noise variance is given by:

$$Var\{y(k)\} = \sigma_q^2 \int_0^1 \left|NTF(e^{j\pi f})\right|^2 df = \sigma_q^2 A(K). \quad (4)$$

where, σ_q^2 is the variance of $q(k)$ and $A(K)$ is the total output noise power amplification factor. Using Parseval's relation, $A(K)$ can be found in the time domain as [1]

$$A(K) = \sum_{k=0}^{\infty} \left|ntf(k)\right|^2 \triangleq \left\|ntf\right\|_2^2 . \quad (5)$$

where $ntf(k)$ is the impulse response corresponding to $NTF(z)$ and $A(k)$ is the squared two-norm of NTF(z). The $A(K)$ curves of the loop-filter are crucial for the stability analysis of the Δ-Σ modulators. Typical curves for Type II Chebyshev 3rd and 4th order are shown in Figure 2. Chebyshev filters achieve better in-band Signal-to-Noise Ratio and Dynamic Range compared with Butterworth filters of the same order.

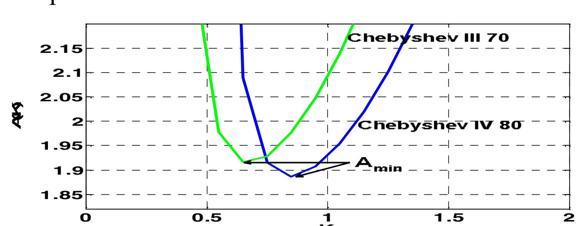

Figure 2. *A(K)* Curves for Type II Chebyshev NTF.

The A_{min} value is the global minimum of the curve. If K increases slightly in the region where $A(K)$ is monotonically increasing, it results in a higher $A(K)$ value which leads to more quantisation noise transfer into the Δ-Σ modulator. This tends to decrease K leading to a stable equilibrium state [1]. However, where the $A(K)$ curve is monotonically decreasing, even small perturbations can destabilize the modulator. As the signal power increases, the values along the $A(K)$ curve decrease and approach A_{min}. The two values of K come close together and finally merge at A_{min}. This characterizes the onset of instability. The modulator operating region escapes to the left portion of the curve where it is characterized by low values of K. Therefore, for stable operation $A(k)>A_{min}$.

III. NOISE AMPLIFICATION CURVES – DF METHOD

The quasilinear quantizer model in Figure 1 can be extended using separate gains K_x and K_n for the DF model as shown in Figures 3 and 4 [4].

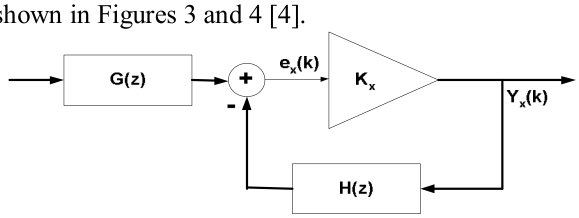

Figure 3. Δ-Σ modulator Quantizer Signal-Model

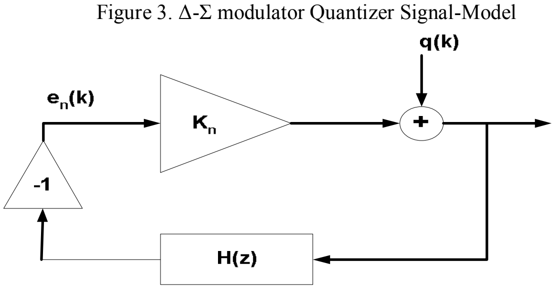

Figure 4. Δ-Σ modulator Quantizer Noise-Model

Figure 3 describes the model for the input signal with linear gain K_x whereas Figure 4 describes the noise signal model with linear gain K_n. The combined output signal is given by:

$$y(k) = y_x(k) + y_n(k). \tag{6}$$

A. DC Input

The linearized gains for a one bit quantizer with an output ±Δ have been calculated in [4] as shown below:

$$K_n = \frac{2\Delta}{\sigma_{en}^2} e^{-m_e^2/2\sigma_{en}^2}. \tag{7}$$

$$K_x = \frac{\Delta}{m_e} erf\left(\frac{m_e}{\sigma_{en}\sqrt{2}}\right). \tag{8}$$

where m_e is the mean value of the quantizer input in the signal model and $\sigma_{e_n}^2$ is the noise variance input to the quantizer in the noise model. The variance of the output signal is given by:

$$Var\{y(k)\} = E\{y^2(k)\} - E^2\{y(k)\}. \tag{9}$$

The output signal in the time-domain can be written as:

$$y(k) = e_n(k)K_n + q(k) + e_x(k)K_x. \tag{10}$$

The first term on the right hand side of (9) is the power of the output signal which is given by:

$$E\{y^2(k)\} = E\{e_n^2(k)K_n^2\} + E\{q^2(k)\} + E\{e_x^2(k)K_x^2\}. \tag{11}$$

$$= \sigma_{en}^2 K_n^2 + \sigma_q^2 + m_e^2 K_x^2. \tag{12}$$

As the mean values of $e_n(k)$ and $q(k)$ are equal to zero, then the second term on the right hand side of (9) is:

$$E^2\{y(k)\} = m_e^2 K_x^2. \tag{13}$$

The resultant variance of the output signal using (9), (12) and (13) becomes:

$$Var\{y(k)\} = \sigma_{e_n}^2 K_n^2 + \sigma_q^2. \tag{14}$$

The noise power amplification factor for a dc input signal $A_{dc}(K)$ after using (4), (7) and (14) simplifies to:

$$A_{dc}(K) = \frac{Var\{y(k)\}}{\sigma_q^2} = \frac{\left(\frac{2}{\pi}\right)\left[e^{-\lambda^2}\right]^2 + \sigma_q^2}{\sigma_q^2}. \tag{15}$$

where λ is a factor defined as: $\lambda = m_e/\sigma_{en}\sqrt{2}$ and σ_q^2 is the quantization noise given by [4]

$$\sigma_q^2 = \Delta^2\left[1 - \frac{m_x^2}{\Delta^2} - \frac{2}{\pi} e^{-2\left[erf^{-1}\left(\frac{m_x}{\Delta}\right)\right]^2}\right]. \tag{16}$$

B. Sinusoidal Input

The linearised gains for a sinusoidal input and random Gaussian feedback components have been solved for the case of an ideal relay in [5] and are shown below:

$$K_n = \left(\frac{2}{\pi}\right)^{\frac{1}{2}}\left(\frac{\Delta}{\sigma_{e_n}}\right) F\left(\frac{1}{2}, 1, -\upsilon^2\right). \tag{17}$$

$$K_x = \left(\frac{2}{\pi}\right)^{\frac{1}{2}}\left(\frac{\Delta}{\sigma_{e_n}}\right) F\left(\frac{1}{2}, 2, -\upsilon^2\right). \tag{18}$$

Here, $\upsilon \Delta \frac{a}{\sqrt{2}} \frac{1}{\sigma_{e_n}}$, where a is the amplitude of the sinusoidal input signal $x(k)$. The expression $F(\alpha, \gamma, x)$ is the confluent hypergoemetric function defined by [6]:

$$F(\alpha,\gamma,\chi) \triangleq 1+\frac{\alpha\chi}{\gamma}+\frac{\alpha(\alpha+1)\chi^2}{\gamma(\gamma+1)\Gamma 2}+\ldots \qquad (19)$$

The variance of the output signal is given by:

$$Var\{y(k)\} = E\{y^2(k)\} - E^2\{y(k)\}. \qquad (20)$$

The power of the output signal is given by:

$$E\{y^2(k)\} = E\{e_n^2(k)K_n^2\} + E\{q^2(k)\} + E\{e_x^2(k)K_x^2\}. \qquad (21)$$

$$= \sigma_{e_n}^2 K_n^2 + \sigma_{q_s}^2 + \sigma_{e_x}^2 K_x^2. \qquad (22)$$

where, σ_{qs}^2 is the quantization noise power for a sinusoidal input. The second term on the right hand side of (20) is:

$$E^2\{y(k)\} = E^2\{e_n(k)K_n\} + E^2\{q(k)\} + E^2\{e_x(k)K_x\} \qquad (23)$$

$$= E^2\{e_x(k)\}K_x^2, \qquad (24)$$

where the mean values of $e_n(k)$ and $q(k)$ are zero. Since the input signal is a sinusoid modelled as a Random Variable (RV) with a certain amplitude and phase having a uniform Probability Density Function (PDF). Therefore, $E\{e_x(k)\}=0$.

$$Var\{y(k)\} = \sigma_{q_s}^2 + K_n^2\sigma_{e_n}^2 + \sigma_{e_x}^2 K_x^2. \qquad (25)$$

Given that the frequency of $x(k)$ is small in the baseband region, this then results in:

$$\frac{E_x(z)}{X(z)} \approx \frac{1}{K_x}. \qquad (26)$$

The variance of $e_x(k)$ is:

$$\sigma_{e_x}^2 = \frac{1}{K_x^2}\sigma_x^2. \qquad (27)$$

From (25) and (27), the output signal variance is:

$$Var\{y(k)\} = \sigma_{q_s}^2 + K_n^2\sigma_{e_n}^2 + \sigma_x^2. \qquad (28)$$

The output noise variance is therefore:

$$Var\{y(k)\} = \sigma_{q_s}^2 + K_n^2\sigma_{e_n}^2. \qquad (29)$$

Substituting (17) in (29), the noise amplification factor for a sinusoidal input signal is given by:

$$A_{\sin e}(K) = \frac{\left(\frac{2}{\pi}\right)F^2\left(\frac{1}{2},1,-\upsilon^2\right)+\sigma_{q_s}^2}{\sigma_{q_s}^2}. \qquad (30)$$

The values of υ and σ_{qs}^2 can be found using the following expressions derived in [4]:

$$\upsilon^2 F^2\left(\frac{1}{2},2,-\upsilon^2\right) = \frac{\pi}{4}\left(\frac{a^2}{\Delta^2}\right). \qquad (31)$$

$$\sigma_{q_s}^2 = \Delta^2\left[1-\frac{a^2}{2\Delta^2}-\frac{2}{\pi}F^2\left(\frac{1}{2},1,-\upsilon^2\right)\right]. \qquad (32)$$

IV. RESULTS & SIMULATIONS

The variation of the dc and sinusoidal input quantisation noise σ_q^2 and σ_s^2 with respect to the input signal amplitude using (16) and (32) are shown in Figure 5. As can be seen, σ_q^2 decreases and becomes zero as the input signal amplitude increases to unity. The quantization noise σ_s^2 does not decrease to zero and remains at 0.3 for an input amplitude of 1.0.

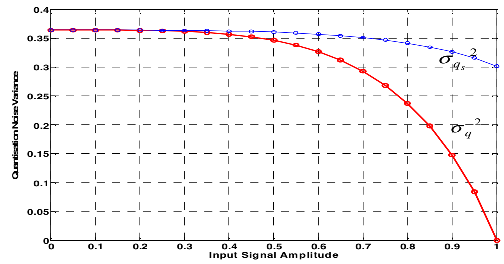

Figure 5. Quantization Noise for dc & Sinusoidal Inputs

Equation (31) has been solved for υ up to the 10th power of the polynomial using a MATLAB routine. Figure 6 shows the variation of λ and υ with respect to the input signal amplitude.

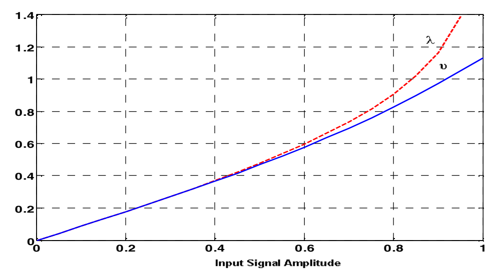

Figure 6. Variation of υ & λ.

It has been observed that for amplitudes less than 0.4, the quantization noise, λ and υ are almost the same for dc and sinusoidal inputs. This coincides with the fact that in nonlinear feedback systems, the effective gain of the non-linearity on a small signal is independent of the type of signal [2]. The noise amplification factors $A_{dc}(K)$ and $A_{sin}(K)$ using (15) and (30) are illustrated in Figure 7. It is seen that the values of $A_{dc}(k)$ using the DF method are the same as in [1].

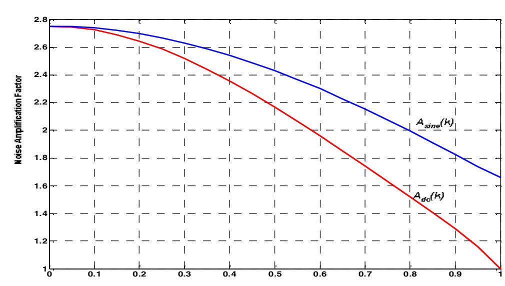

Figure 7. Noise Amplification Factor for sinusoidal & dc inputs

Using $A_{dc}(k)$ and $A_{sine}(k)$, the maximum stable input amplitudes for the 3rd, 4th and 5th order Chebyshev Type II based Δ-Σ modulator are demonstrated in Figure 8.

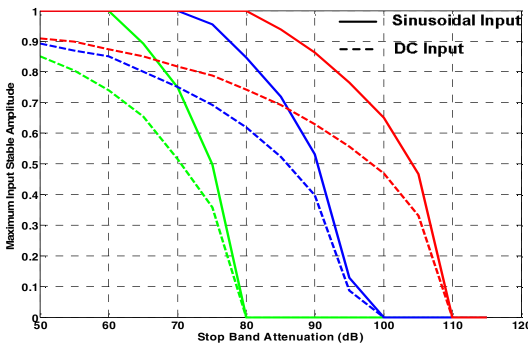

Figure 8. Maximum stable input amplitude for Chebyshev Type II NTF(z).

However, these are true for unity values of K. The variation of the stable sinusoidal input amplitude for a 4th-order Chebyshev Type II based Δ-Σ modulator in relation to K and the stop-band attenuation is shown in Figure 9.

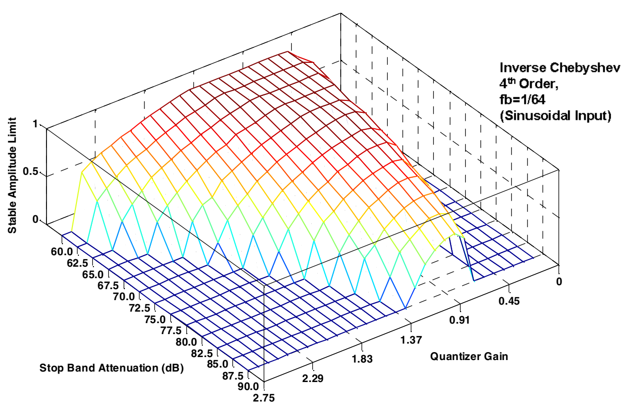

Figure 9. Stable amplitude variation with Quantizer Gain & stop band.

The stable input amplitude variation for dc and sinusoidal inputs to a 5th order Chebyshev Type II based Δ-Σ modulator (stop band attenuation at 67 dB) is shown in Figure 10.

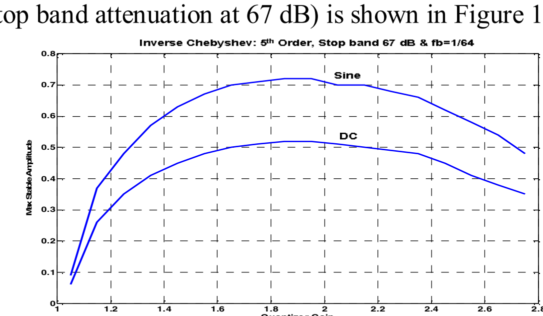

Figure 10. Stable amplitude variation with Quantizer Gain

Simulations for the 5th-order Chebyshev Type II based Δ-Σ modulator implemented in the feed forward topology were undertaken for 1638400 time samples with an increase in input amplitude in steps of 0.1. The maximum stable amplitude limits were observed and the corresponding values of K were calculated using (33) [7]

$$K = Covariance\{e(k), y(k)\}/\sigma_e^2 \quad (33)$$

The predicted values of the maximum stable amplitudes were obtained from Figure 10.

Table I. SIMULATION RESULTS.

Signal	K	Predicted Stable Amplitude	Stable Amplitude As per Simulations
dc	1.62	0.52	0.63
sine	1.70	0.69	0.66

The difference in values is attributed to the composition of the quantization noise which is not entirely Gaussian [7].

V. CONCLUSION

The stability of higher-order Δ-Σ modulators for dc and sinusoidal inputs using the Describing Function Method has been predicted. The maximum stable input limits for the 3rd-, 4th- and 5th-order Chebyshev Type II based Δ-Σ modulator have been established for a unity quantizer gain. More accurate results for the stable amplitude curves can be obtained for a range of values of quantizer gain K in which the Δ-Σ modulators are likely to operate. A future publication would include the analysis for multiple sinusoidal inputs.

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