Accurate stability prediction of single-bit higher-order delta-sigma (Δ-Σ) modulators for Speech Codecs

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***Abstract*— Present approaches for predicting the stability of Delta-Sigma (Δ-Σ) modulators are mostly confined to DC inputs. This poses limitations as practical applications of Δ-Σ modulators involve a wide range of signals other than DC such as speech, audio and multiple sinusoids as is the case for synthetically generated speech or audio or any other signal, through the appropriate composition of multiple sinusoids. In this paper, a quasi-linear model that accurately predicts stability of single-loop 1-bit higher-order Δ-Σ modulators for multiple sinusoids is given. The results of this paper would enable optimization of the design of higher-order single-loop Δ-Σ modulators with increased dynamic ranges for various applications that deploy multiple sinusoidal inputs, as well as any general input signal constructed from multiple sinusoids.**

I. Introduction

The stable input amplitude limits for Delta-Sigma (Δ-Σ) modulators are complicated to predict due to the non-linearity of the quantizer. The stable input amplitude limit decreases as the order of the Δ-Σ modulator increases. One technique is to model the quantizer as a threshold function in state equations, which gets complicated for higher-order Δ-Σ modulators and is limited to 1^{st}- and 2^{nd}- order Δ-Σ modulators [1]. Another approach has been to assume a DC input to the Δ-Σ modulator [2]-[7]. The linearised modeling approach in [2] did not previously provide useful stability predictions until a new interpretation of the instability mechanism for Δ-Σ modulators based on the Noise Amplification Factor was given in [8]. However, this was restricted to DC inputs only. The analysis was extended in [9], where the stability predictions based on the Noise Amplification Factor, were given for single-sinusoidal and dual-sinusoidal inputs.

In this paper, a quasi-linear model of the Δ-Σ modulator that deploys nonlinear feedback control analysis is able to accurately predict the stability of 1-bit higher-order Δ-Σ modulators for multiple sinusoidal inputs. Simulations indicate accurate results from this novel model, which would enable optimization of the design of Δ-Σ modulators with increased dynamic ranges for a range of applications including synthetic speech codecs. Section II describes the stability mechanism in terms of the quasi-linear model and Noise Amplification Factor. Section III describes the ratio of the signal variance to the quantization noise variance at the quantizer input followed by variation of quantization noise in the Δ-Σ modulator. Section IV establishes the variation of Noise Amplification Factor with the input signal amplitude. Simulations are given in Section V, followed by conclusions in Section VI.

II. Quasi-linear Δ-Σ Modulator and Noise Ammplification Factor

A quasi-linear model of a Δ-Σ modulator is shown in Fig.1, where $G(z)$ is the input transfer function, $H(z)$ the feedback filter transfer function and the quantizer is replaced by a gain factor K followed by an additive white quantization noise source $q(n)$:

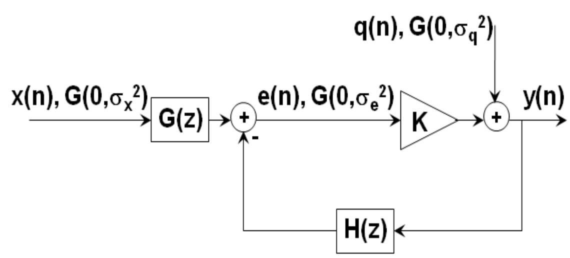

Fig. 1. Quasi-linear Δ-Σ modulator.

Assuming $q(n)$ to be white with zero mean and variance σ_q^2, and the Noise Transfer Function (*NTF*) between $q(n)$ and $y(n)$ to be known, the noise variance V_o at the output of the Δ-Σ modulator is given by [8]:

$$V_o = \sigma_q^2 \int_0^1 \left|NTF(e^{j\pi f})\right|^2 df = \sigma_q^2 A(K) \qquad (1)$$

where, $A(K)$ is the Noise Amplification Factor. Using Parseval's theorem, $A(K)$ can also be found in the time-domain as [8]:

$$A(K) = \sum_{n=0}^{\infty} |ntf(n)|^2 \triangleq \|ntf\|_2^2 \qquad (2)$$

where $ntf(n)$ is the impulse response corresponding to $NTF(z)$ and $\|ntf\|_2^2$ is the squared second-norm of $ntf(n)$. The variation of $A(K)$ with K can be plotted as $A(K)$ curves from (2) and is used to explain the stability of Δ-Σ modulators. A typical curve for a 4th-order Δ-Σ modulator is shown in Fig. 2, where A_{min} is the global minimum value of the curve which is 2.41.

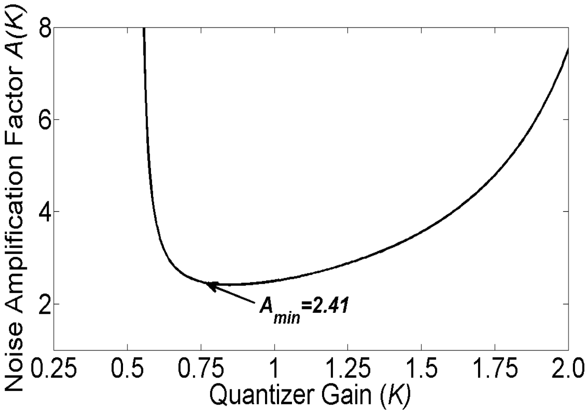

Fig. 2. Noise Amplification Factor variation with quantizer gain.

For stable operation of the Δ-Σ modulator, $A(K) > A_{min}$.[8]. If one can quantify the variation of $A(K)$ with the input signal amplitude a for which $A(K) > A_{min}$, then the maximum input stable amplitude limits can be established. Quantifying $A(K)$ is done in two steps, first by estimating the ratio of signal variance to quantization noise variance at the input to the quantizer gain K and subsequently estimating the variation of quantization noise variance σ_q^2 with the input signal amplitude in the Δ-Σ modulator loop.

III. Signal and Noise Variance At Quantizer Input and Quantization Noise

A. *The ratio of the signal variance to quantization noise variance at the quantizer input*

An input signal, consisting of five incommensurate sinusoids with variances σ_i^2, $i=1,2,\ldots 5,$ has a Gaussian distribution with variance σ_x^2 and is given by:

$$\sigma_x^2 = \sum_{i=1}^{5} \sigma_i^2 \qquad (3)$$

If σ_{ex}^2 and σ_{eq}^2 are variances of the signal and quantization noise at the input to the quantizer, the combined variance at the quantizer input is given by:

$$\sigma_e^2 = \sigma_{ex}^2 + \sigma_{eq}^2 \qquad (4)$$

The quantizer gain K of a Δ-Σ modulator for a Gaussian input with variance σ_e^2 for a single-bit output $\pm\Delta$ is given by [10]:

$$K = \frac{\Delta}{\sigma_e}\sqrt{\frac{2}{\pi}} \qquad (5)$$

Defining ρ^2 as the ratio of the signal variance to the quantization noise variance at the quantizer input yields the equation below:

$$\rho^2 = \frac{\sigma_{ex}^2}{\sigma_{eq}^2} \qquad (6)$$

The quantizer gain K can be found from (4), (5) and (6) as given by:

$$K = \frac{\Delta}{\sigma_{eq}} \frac{1}{\sqrt{1+\rho^2}} \sqrt{\frac{2}{\pi}} \qquad (7)$$

The Signal Transfer Function (*STF*) of the Δ-Σ modulator should ideally be ≈ 1, therefore:

$$K^2 \sigma_{ex}^2 = \sigma_x^2 \qquad (8)$$

As Δ can be assumed to be equal to 1 for a single-bit quantizer, one gets the following relation from (7) and (8):

$$\frac{2}{\pi}\left[\frac{\sigma_{ex}^2}{\sigma_{eq}^2}\right]\frac{1}{[1+\rho^2]} = \sigma_x^2 = \frac{1}{2}a^2 \qquad (9)$$

$$\Rightarrow \frac{2}{\pi}\rho^2 \frac{1}{[1+\rho^2]} = \frac{1}{2}a^2 \qquad (10)$$

By solving (10), we get ρ which is plotted in Fig. 3. The variation of ρ with a is linear for $a \leq 0.5$ and becomes nonlinear for $a > 0.5$.

B. *Quantization Noise Variance*

If $E(.)$ is the expectant operator, the power at the output of the Δ-Σ modulator is given by:

$$E[y^2(n)] = \sigma_q^2 + K^2\sigma_{ex}^2 + K^2\sigma_{eq}^2 = \Delta^2 \qquad (11)$$

The quantization noise variance σ_q^2 can be found from (7), (8) and (11):

$$\sigma_q^2 = \Delta^2\left[1 - \frac{a^2}{2\Delta^2} - \left(\frac{2}{\pi}\right)\frac{1}{[1+\rho^2]}\right] \qquad (12)$$

Assuming Δ as ±1 for the single-bit quantizer, the quantization noise variance σ_q^2 is obtained from (12) and is plotted as in Fig. 3. The quantization noise variance remains constant for $a \leq 0.6$ and decreases for $a > 0.6$.

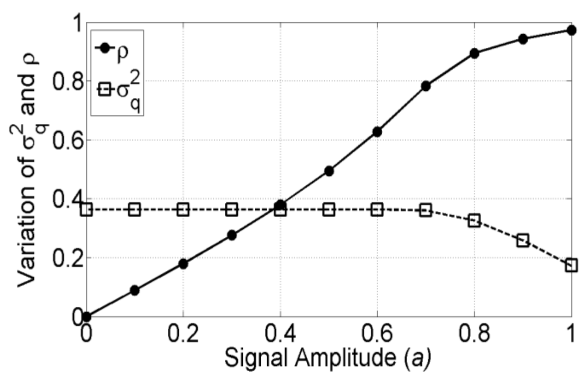

Fig. 3. Variation of signal variance to quantization noise variance at the quantizer input; variation of quantization noise with signal input amplitude.

IV. Variation of Noise Amplification Factor

The noise variance at the output of a Δ-Σ modulator is:

$$V_o = \sigma_q^2 + K^2\sigma_{eq}^2 \qquad (13)$$

From (7) and (13) we get:

$$V_o = \sigma_q^2 + \frac{2}{\pi}\frac{1}{1+\rho^2} \qquad (14)$$

The variation of $A(K)$ with a can be found by (1) and (14), which is given as:

$$A(K) = \frac{V_o}{\sigma_q^2} = \frac{\sigma_q^2 + \frac{2}{\pi}\left[\frac{1}{1+\rho^2}\right]}{\sigma_q^2} \qquad (15)$$

Using (15), $A(K)$ is plotted as in Fig. 4. $A(K)$ decreases as a increases and reaches A_{min} at which point the Δ-Σ modulator becomes unstable. From (1), it is seen that as the Δ-Σ modulator order increases, so does V_o, thereby increasing A_{min} in Fig. 2. Thus, the Δ-Σ modulator gets unstable at lower amplitudes for a higher *NTF* order. On the other hand, from (15) it is seen that as the number of quantizer bits increases, the quantization noise σ_q^2 decreases, thereby maintaining $A(K)$ in Fig. 4 at a higher level for the same input amplitude a as compared to the 1-bit quantizer and therefore increasing the input stable amplitude limit for the same *NTF* order.

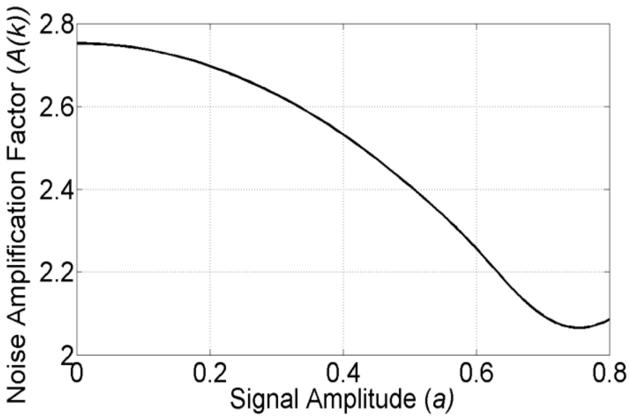

Fig. 4. Variation of Noise Amplification Factor with signal input amplitude.

V. SIMULATION RESULTS

Simulations were undertaken for 4th-order single-loop single-bit Δ-Σ modulators for the corresponding $A(K)$ curve in Fig. 2. The Δ-Σ modulator is implemented by deploying a Cascade-of-Accumulators FeedBack-form (CAFB) topology as shown in Fig. 5.

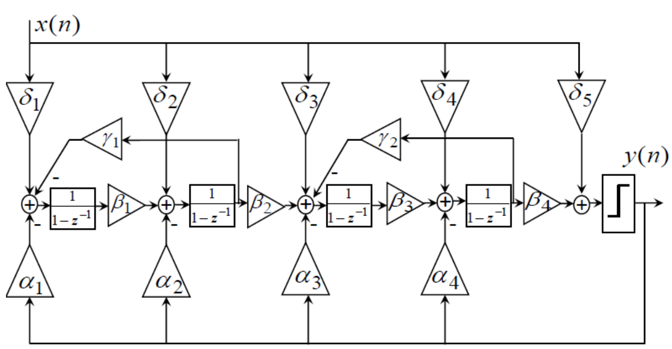

Fig. 5. Fourth-order Δ-Σ modulator in CAFB topology.

The coefficient values for the fourth-order Δ-Σ modulator are shown in Table I.

TABLE I
COEFFICIENTS FOR 4TH-ORDER Δ-Σ MODULATOR

Δ Σ	I	1	2	3	4	5
4th-order	δ_i	0.0157	0.1359	0.5140	0.3609	1.0000
	α_i	0.0157	0.1359	0.5140	0.3609	-
	β_i	1.0000	1.0000	1.0000	1.0000	-
	γ_i	0.003	0.0018	-	-	-

The input signal consists of five incommensurate sinusoid amplitudes, which are increased in steps of 0.0003 that increases the overall amplitude a. The sinusoids are selected at random frequencies with a maximum frequency of 7.8 kHz. The Δ-Σ modulator clock frequency is 500 kHz, which results in an Over-Sampling Ratio (OSR) of 32 for 25×10^5 output samples. The Spurious-Free Dynamic Range (SFDR) is obtained by plotting the periodogram Power Spectral Density (PSD) using a Hanning window, which is useful for measuring random signals and offers good frequency resolution. The variation in the SFDR with respect to a for the Δ-Σ modulator of Fig. 5. is shown in Fig. 6.

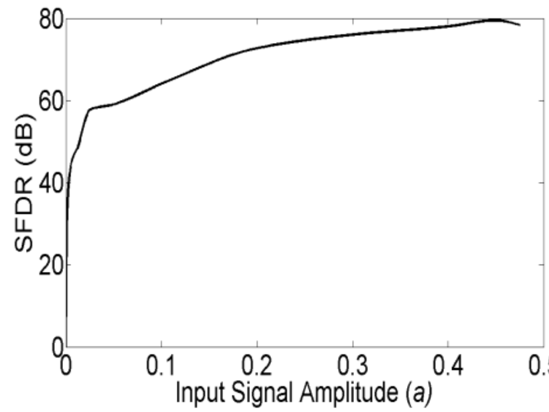

Fig. 6. 4th-order variation of SFDR with input signal amplitude.

The SFDR increases linearly with an increase in a. At a = 0.48, it starts to fall showing the onset of instability. The numerical stable value of a as predicted from Fig. 4 is 0.49, when $A(K) = 2.41 = A_{min}$. As long as the sum of the five sinusoids $a < 0.48$, the 4th-order Δ-Σ modulator is stable, irrespective of the individual sinusoidal amplitude levels.

For any *NTF,* one can plot the $A(K)$ curve and accurately predict stability of the Δ-Σ modulator from Fig. 4 circumventing the need for detailed simulations. One can therefore optimize the Δ-Σ modulator parameters to be deployed in a synthetic speech codec deriving the synthetic speech signals from the suitable composition of five individual sinusoids as depicted in Fig. 7, and as is the case with LPC10 [11] that is in many applications fed into a Δ-Σ modulator based DAC to generate the actual analogue speech.

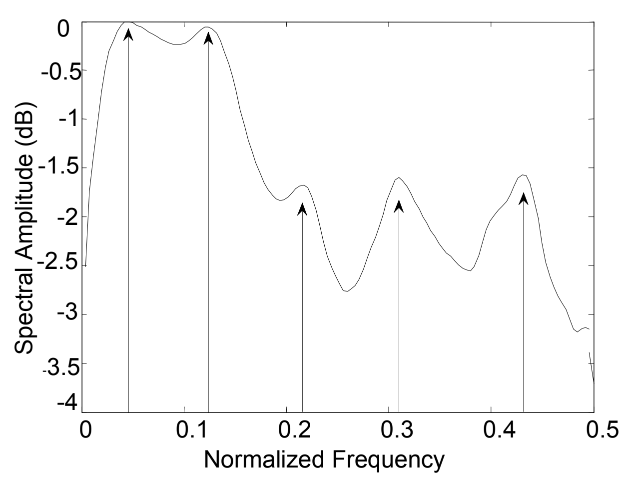

Fig. 7. Spectral Plot for the Word 'Five' Synthesized through LPC10.

VI. CONCLUSIONS

The stable input amplitude limits have been predicted accurately for single-bit 4^{th}- Δ-Σ modulator for five sinusoidal inputs. The theoretical values are shown to be in very close agreement with the simulation results. The analysis can be extended to any number of sinusoids greater than five, for any *NTF*, be it low-pass, high-pass or band-pass or any other. The Δ-Σ modulator relationship between stability, increase in the *NTF* order and number of quantizer bits has been mathematically explained and novel results are reported. The novel results would enable optimizing the design of higher-order Δ-Σ modulators for various applications that require multiple sinusoidal inputs such as speech and moreover for any general test or other inputs that can be modeled as the Fourier series of individual sinusoids.

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