SETTLING TIME FORMULAE FOR THE DESIGN OF CONTROL SYSTEMS WITH LINEAR CLOSED LOOP DYNAMICS

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Abstract: Two settling time formulae are numerically derived with the 5% and 2% criteria for the step responses of control systems having linear closed loop dynamics that may be designed by the method of pole assignment to have multiple closed loop poles. The formula is shown to be accurate for closed loop systems of up to tenth order. To clarify the use of the formulae, model based and robust control system designs are carried out for a high precision vacuum air bearing application and experimental results presented.

1. Introduction

It is well known that the classical approach to control system design is based in the frequency domain with the assumption of linear plant dynamics and for single input, single input plants is a linear controller which is either one of several variations on the proportional integral derivative (PID) theme or a compensator. This approach is well documented in text books and has been established in industry for many years. Its evolution, however, has been influenced greatly by the constraints of practicable and economical hardware implementation of the past, the first versions being specially tailored analogue electronic circuits implemented with discrete components, the active components initially being thermionic valves, later replaced by transistors and more recently operational amplifiers. Such circuits are now being replaced by digital processors such as microcontrollers, field programmable gate arrays (FPGA) and digital signal processors (DSP), which are specially designed for fast execution of relatively heavy computational loads. The cost of control hardware is now drastically reduced by the software implementation of specific controllers on readily available and mass produced processors. This allows sophisticated control techniques yielding better performance than traditional controllers to be applied economically. Even today, however, the structure of many digitally implemented controllers follows the aforementioned classical tradition and sometimes leads to compromised performance in specific applications. This paper contains examples of control techniques that take full advantage of the digital implementation media in catering for plants that may be nonlinear and of arbitrarily high order, the control design being rendered especially straightforward and palatable by students of control engineering and control engineers in industry by the settling time formulae presented. In fact, the author derived a step response settling time formula for the 5% criterion which is already in use, exemplified by Vittek and Dodds (2003) and this paper formally presents this together with a formula for the 2% criterion, which is new.

The standard settling time formulae for the step responses of first and second order linear control systems using the 5% criterion are well documented in text books but in many cases, the system order is greater than 2, often 3 and not infrequently extending to 6. Occasionally, much higher orders are encountered in applications such as active vibration control and for this reason, systems up to 20th order are considered.

Linear control theory is well understood and therefore the design of a control system is made relatively simple if a control technique is used that renders the closed loop system linear. This paper is restricted to single input, single output plants but a similar approach may be taken for the control of multivariable plants. The design approach presented here is a simple 'top-down' one in which the starting point is the desired performance in terms of the settling time with zero overshoot in the step response. It is important to note that if control energy is an important factor, then the settling time can be adjusted to minimise this within the constraint of maximum allowable settling time, attention also being paid to robustness (external disturbance rejection and insensitivity to plant modelling errors).

It will be assumed here that the sampling time of the digital processor is sufficiently small compared with the time constants and modal periods of the open and closed loop system for continuous control theory to be applicable. The methods presented here are, however, extendable to discrete control theory.

A non-overshooting step response is a good starting point in the time domain for most control system designs and this can be achieved if the closed loop system has coincident negative real poles, according to the transfer function:

$$\frac{y(s)}{y_r(s)} = \left(\frac{1}{1+sT_c}\right)^n \qquad (1.1)$$

where y is the measured and controlled output, y_r is the reference input, and T_c is the closed-loop time constant, i.e., the time constant of each of the n first order systems which when connected in a chain will yield the desired n^{th} order closed loop dynamics. Note that the usual unity d.c. gain is assumed but this can be made different if required.

It is important to note that a controller must be used containing at least n adjustable parameters that can be set to realise the desired closed loop dynamics defined by (1.1). The flexibility of modern digital implementation renders this possible for all controllable plants.

2. Derivation of Formulae

2.1. Formulation of the problem

Let y_{ss} be the steady state response to a step reference input. Then the settling time according to the x% criterion is defined as the time taken for $|y_{ss} - y(t)|$, to reduce to and thereafter remain less than x% of its maximum value. Traditionally x is chosen as 2 or 5 and therefore only these criteria are considered in this paper.

Fig. 2.1 shows a family of step responses of the closed loop system defined by (1.1) for orders ranging between 1 and 20. The step reference input is $y_r(t) = Y_r h(t)$, where $h(t)$ is the Heaviside unit step function. The outputs are normalised with respect to Y_r, i.e., $y'(t) = y(t)/Y_r$ and the time is normalised with respect to T_c, i.e., $t' = t/T_c$.

The settling times for the 5% and 2% criteria are, respectively, the times taken for the normalised step responses to cross the horizontal straight lines, $y' = 0.95$ and $y' = 0.98$. A formula relating the settling time to the order for the closed loop system defined by (1.1) would be a valuable tool for the design of controllers using the aforementioned top-down approach.

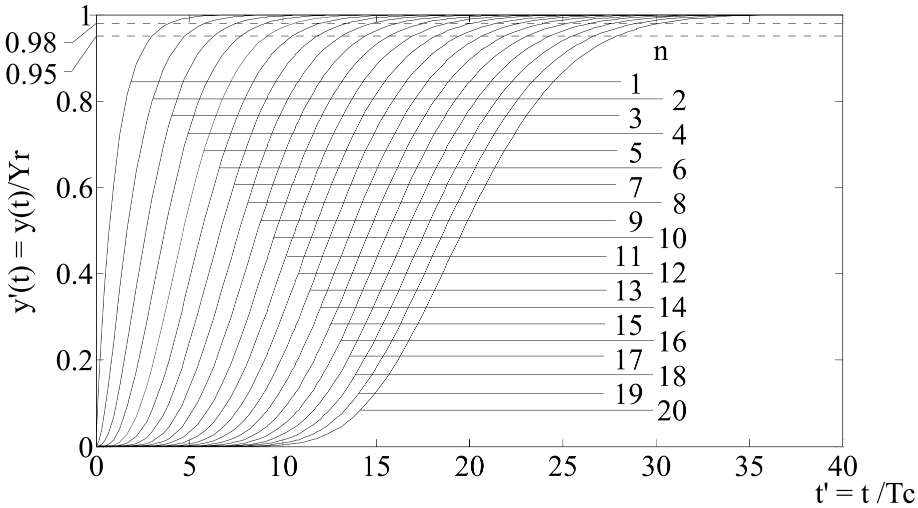


Fig. 2.1: Family of normalised step responses of linear closed loop system with multiple poles.

The mathematical formulation of the problem of deriving such a formula is as follows. It may be shown that the general expression of the step response of the closed loop system defined by (1.1) is:

$$y(t) = Y_r\left[1-\sum_{i=0}^{n-1}\frac{1}{i!}\left(\frac{t}{T_c}\right)^i e^{-\frac{t}{T_c}}\right] \quad (2.1)$$

which may be written as follows in terms of the normalised quantities already defined:

$$y'(t) = \left[1-\sum_{i=0}^{n-1}\frac{1}{i!}(t')^i e^{-t'}\right] \quad (2.2)$$

The normalised settling time for the x% criterion would therefore satisfy

$$y'(T'_{sx\%}) = \left[1-\sum_{i=0}^{n-1}\frac{1}{i!}(T'_{sx\%})^i e^{-T'_{sx\%}}\right]$$ and

since $x = 100\left[1-y'(T'_{sx\%})\right]$, then

$$0.01x = \sum_{i=0}^{n-1}\frac{1}{i!}(T'_{sx\%})^i e^{-T'_{sx\%}} \quad (2.3)$$

The *exact* settling time formula would be given by the closed form solution to (2.3):

$$T'_{sx\%} = f(n,x) \Rightarrow T_{sx\%} = \left[f(n,x)\right]T_c \quad (2.4)$$

The exact function, $f(n,x)$, however, has not yet been discovered, but section 2.2 provides a practicable approximation.

2.2. A numerical solution

Observation of Fig. 2.1 reveals that the differences between the settling times of the responses of systems differing in order by 1 are nearly equal, so a linear approximation to the function, $f(n,x)$ is possible for $x = 2$ and 5. For this purpose, the normalised settling times have been precisely computed for $n = 1,2,\ldots,20$ using a Matlab Simulink variable step simulation and Table 2.1 shows the results. These are plotted in Fig. 2.2 (a) and (b) for $x = 5$ and $x = 2$, respectively. The question now arises of the choice of the linear approximation method and how many points to take. The classical approach would be to use all 20 points and apply a least squares fit.

Due to the progressive reduction in slope of the graphs evident in Fig. 22, however, the maximum and minimum approximation errors increase in magnitude with the number of points. Judgement has therefore been exercised to restrict the number of points to include the most encountered system orders

Table 2.1: Normalised settling times of linear closed loop system with multiple poles.

Order, n	$T'_{s5\%} = T_{s5\%}/T_c$	$T'_{s2\%} = T_{s2\%}/T_c$
1	2.9957	3.9118
2	4.7438	5.8338
3	6.2958	7.5165
4	7.7537	9.0841
5	9.1536	10.5804
6	10.5131	12.0270
7	11.8424	13.4364
8	13.1482	14.8166
9	14.4347	16.1731
10	15.7053	17.5098
11	16.9623	18.8298
12	18.2075	20.1352
13	19.4426	21.4279
14	20.6686	22.7094
15	21.8865	23.9809
16	23.0972	25.2434
17	24.3012	26.4976
18	25.4992	27.7444
19	26.6918	28.9844
20	27.8793	30.2181

to obtain better approximations for these than would be obtained by using all the points. With reference to Table 2.1, the formula should fit the well known result of $T_{s5\%} = 2.9957T_c \cong 3T_c$ for $n = 1$, so the point $[n, T_{s5\%}] = [1, 3]$ should be one point on the straight line fit for $x = 5$. It may also be observed that for $n = 1$, $T_{s2\%} = 3.9118T_c \cong 4T_c$. This approximation is not quite as accurate as that for $x = 5$, but is chosen to yield a simple formula so the point $[n, T_{s2\%}] = [1, 4]$ will be chosen for $x = 2$. As reasoned in section 1, a practical approach would be to consider points up to $n = 6$ and observation of Table 2.1 reveals two more points through which the straight line approximations can pass that should yield simple formulae. Thus, for $n = 6$, $T_{s5\%} = 10.5131T_c \cong 10.5T_c$ yielding the point $[n, T_{s5\%}] = [6, 10.5]$ and $T_{s2\%} = 12.0270T_c \cong 12T_c$ yielding the point $[n, T_{s2\%}] = [6, 12]$.

The straight line fit

$$T'_{sx\%} = C_x + M_x n \qquad (2.5)$$

where C_x and M_x are constants to be determined will be applied to the aforementioned fixing points in section 2.3.

2.3. The 5% settling time formula

Using the fixing points of Figure 2.2 (a) yields:

$$\begin{cases} C + M = 3 \\ C + 6M = 10.5 \end{cases} \Rightarrow \begin{cases} 5M = 7.5 \Rightarrow M = 3/2 \\ C = 3 - M = 3/2 \end{cases} \Rightarrow$$

$T'_{s5\%} = 3(1+n)/2$. The settling time formula for the 5% criterion is therefore as follows:

$$\boxed{T_s = \tfrac{3}{2}(1+n)T_c} \text{ or } \boxed{T_s = 1.5(1+n)T_c} \qquad (2.6)$$

2.4. The 2% settling time formula

Using the fixing points of Figure 2.2 (b) yields:

$$\begin{cases} M + C = 4 \\ 6M + C = 12 \end{cases} \Rightarrow \begin{cases} 5M = 8 \Rightarrow M = 8/5 \\ C = 4 - M = 12/5 \end{cases} \Rightarrow$$

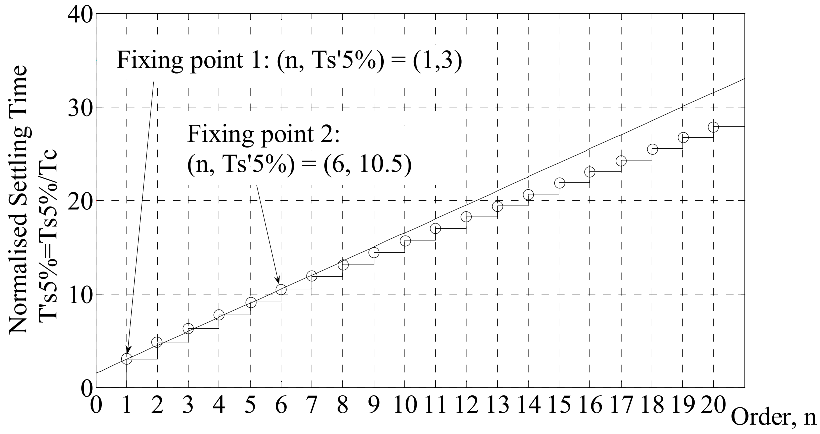
$T'_{s2\%} = \frac{4}{5}(3+2n)$. The settling time formula for the 2% criterion is therefore as follows:

$$\boxed{T_s = \tfrac{4}{5}(3+2n)T_c} \text{ or } \boxed{T_s = 1.6(1.5+n)T_c} \qquad (2.7)$$

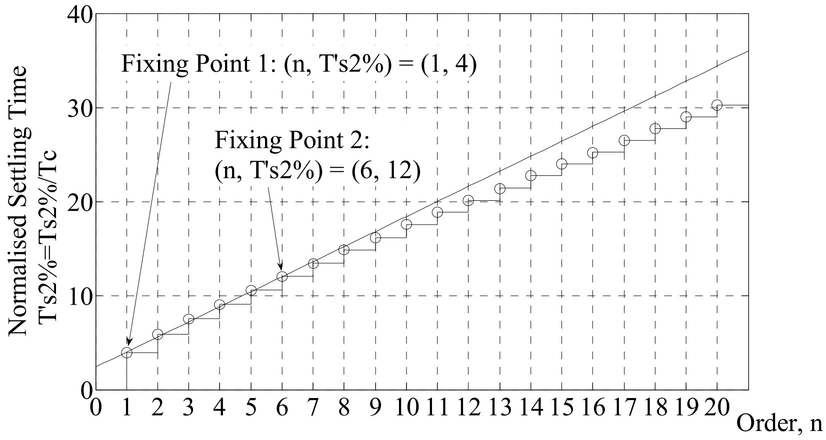
3. Accuracy Assessment

The settling times, T_s, obtained using nominal settling times, $T_{s\,nom}$, in (2.6) and (2.7) were accurately determined by means of variable step Matlab-Simulink simulations.

The normalised settling times, presented in Table 3.1 are $\tilde{T}_s = T_s/T_{s\,nom}$ and would therefore be unity without errors. The percentage errors shown are therefore calculated as $100(\tilde{T}_s - 1)\%$.



a) 5% criterion



b) 2% criterion

Fig. 2.2: Normalised settling times and straight line fits

For $n = 1, 2, \ldots, 10$, the errors are within $\pm 5\%$. This is considered acceptable for most control system designs, as illustrated by the families of step responses in Fig. 2.3, which all pass nearly through the point for which $t = T_s$. If higher accuracy is required, however, then if the measured settling time is T_{sm}, then a 'single iteration' correction could be made by simply setting T_c to a new value:

$$T_{c\,new} = \left(T_{s\,nom} / T_{sm} \right) T_c \qquad (2.8)$$

where $T_{s\,nom}$ is the required settling time. Alternatively, Table 2.1 could be used to yield the value of T_c needed to realise the specified value of $T_{sx\%}$, $x = 2$ or 5, and this applies to all system orders. According to Table 3.1, even for $n = 20$, the actual settling time, T_s, would be approximately 12% less than the specified one, T_{nom}, using (2.6) or (2.7) and the correct settling time could still be realised by adjusting T_c as described above.

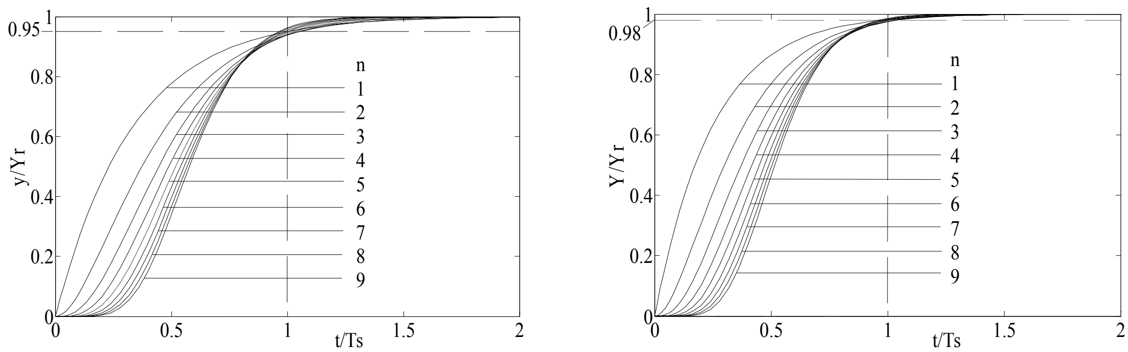
4. Control System Design Examples

4.1 The plant

The plant is a single axis vacuum air bearing rig, used by Stadler et. al. (2005), consisting of a slider of mass, M, with position, x, along a horizontal friction free guide and controlled by a voice coil actuator with control voltage, u, the position measurement, y, coming from a high resolution encoder.

Table 3.1: Normalised settling times and % errors

n	5% Criterion		2% Criterion		n	5% Criterion		2% Criterion	
	$\tilde{T}_s$	% err.	$\tilde{T}_s$	% err.		$\tilde{T}_s$	% err.	$\tilde{T}_s$	% err.
1	0.9986	–0.14	0.9780	–2.20	11	0.9423	–5.77	0.9415	–5.85
2	1.0542	+5.42	1.0418	+4.18	12	0.9337	–6.63	0.9322	–6.78
3	1.0493	+4.93	1.0440	+4.40	13	0.9258	–7.42	0.9236	–7.64
4	1.0338	+3.38	1.0323	+3.23	14	0.9186	–8.14	0.9157	–8.43
5	1.0171	+1.71	1.0173	+1.71	15	0.9119	–8.81	0.9083	–9.17
6	1.0012	+0.12	1.0022	+0.22	16	0.9058	–9.42	0.9015	–9.85
7	0.9869	–1.31	0.9880	–1.20	17	0.9000	–10.00	0.8951	–10.9
8	0.9739	–2.61	0.9748	–2.52	18	0.8947	–10.53	0.8892	–11.08
9	0.9623	–3.77	0.9627	–3.73	19	0.8897	–11.03	0.8836	–11.64
10	0.9518	–4.82	0.9516	–4.84	20	0.8850	–11.50	0.8784	–12.16



a) 5% criterion b) 2% criterion

Fig. 2.3: Normalised step responses for multiple pole placement using the settling time formulae.

The plant equations are:

$$\ddot{x} = f/M, f = K_a u, y = K_m x \Rightarrow \underline{\ddot{y} = bu} \quad (4.1)$$

where $b = K_a K_m / M$. The slider mass, actuator constant and measurement constant are, respectively, $M = 3.25\,kg$, $K_a = 0.8\ A/V$ and $K_m = 11.1\ N/A$.

The application of the settling time formulae in the design of a high precision motion control system for the rig will now be demonstrated, using pole placement first for a cascade IPD controller and then for a sliding mode controller (SMC), ref., Utkin (1992), with a boundary layer and an integral outer loop to remove steady-state errors due to disturbance forces.

4.2 The desired characteristic polynomials

In every case, application of the settling time formulae (2.6) and (2.7) to the desired closed loop transfer function (1.1) yields:

$$\frac{y(s)}{y_r(s)} = \underbrace{\left(\frac{1}{1+s\frac{2T_s}{3(1+n)}}\right)^n}_{\text{a) 5\% criterion}} \text{ or } \underbrace{\left(\frac{1}{1+s\frac{5T_s}{4(3+2n)}}\right)^n}_{\text{b) 2\% criterion}} \quad (4.2)$$

The corresponding desired characteristic polynomials normalised w.r.t. the coefficient of s^n are therefore given by:

$$\underbrace{\left(s+\frac{3(1+n)}{2T_s}\right)^n}_{\text{a) 5\% criterion}} \text{ or } \underbrace{\left(s+\frac{4(3+2n)}{5T_s}\right)^n}_{\text{b) 2\% criterion}} \quad (4.3)$$

4.2 Design of the IPD cascade controller

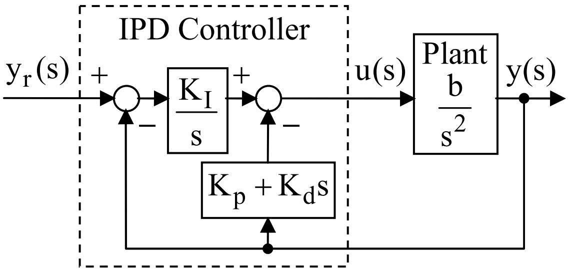


Fig. 4.1: IPD position control system

The closed loop characteristic polynomial is $s^3\Delta(s)$, where $\Delta(s)$ is the determinant of Mason's formula applied to Fig. 4.1:

$$s^3\left\{1-\left[-\frac{b}{s^2}\left(K_d s+K_p+\frac{K_I}{s}\right)\right]\right\}=$$

$$s^3+bK_d s^2+bK_p s+bK_I \quad (4.4)$$

For $n=3$, (4.3) yields:

$$\left(s+\frac{6}{T_s}\right)^3=s^3+\frac{18}{T_s}s^2+\frac{108}{T_s^2}s+\frac{216}{T_s^3} \quad (4.5)$$

for the 5% criterion and

$$\left(s+\frac{36}{5T_s}\right)^3=s^3+\frac{108}{5T_s}s^2+\frac{1296}{25T_s^2}s+\frac{46656}{125T_s^3} \quad (4.6)$$

for the 2% criterion. Comparing (4.4) with (4.5) and (4.6) in turn yields the required controller gains for the chosen criterion:

$$K_d^{5\%}=\frac{18}{T_s b},K_p^{5\%}=\frac{108}{T_s^2 b},K_I^{5\%}=\frac{216}{T_s^3 b} \quad (4.7)$$

$$K_d^{2\%}=\frac{108}{5T_s b},K_p^{2\%}=\frac{1296}{25T_s^2 b},K_I^{2\%}=\frac{46656}{125T_s^3 b}$$

(4.8)

4.3 Design of the sliding mode controller

Referring to Fig. 4.2, the sliding function, $S(y,\dot{y},y_r)$ is linear and driven to zero in the sliding mode so that the closed loop

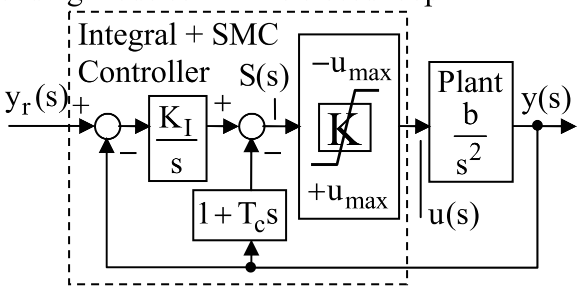


Fig. 4.2: Integral + SMC position control system.

characteristic equation is given by $S(s)=0$ with $y_r(s)=0 \Rightarrow -\left(T_c s+1+\frac{K_I}{s}\right)y(s)=0 \Rightarrow$

$$s^2+\frac{1}{T_c}s+\frac{K_I}{T_c}=0 \quad (4.9)$$

For $n=2$, (4.3) yields:

$$\left(s+\frac{9}{2T_s}\right)^2=s^2+\frac{9}{T_s}s+\frac{81}{4T_s^2} \quad (4.10)$$

for the 5% criterion and

$$\left(s+\frac{28}{5T_s}\right)^2=s^2+\frac{56}{5T_s}s+\frac{784}{25T_s^2} \quad (4.11)$$

for the 2% criterion. Comparing (4.9) with (4.10) and (4.11) in turn yields the required controller parameters for the chosen criterion:

$$T_c^{5\%}=\frac{T_s}{9},K_I^{5\%}=\frac{9}{4T_s},T_c^{2\%}=\frac{5T_s}{56},K_I^{5\%}=\frac{14}{5T_s} \quad (4.12)$$

Note that the plant parameter, b, is not needed, indicating the extreme robustness of SMC.

4.4 Experimental results

The plant hardware is briefly described at the beginning of section 4.1. For both controllers, the settling time is set to $T_s = 0.1\,s$, a step reference position of $10\,\mu m$ is applied and the sampling frequency of the DSpace implementation is 40 kHz. For the SMC, the slope of the transfer characteristic realising the boundary layer is $K = 2x10^4$. Figs 4.3 to 4.6 show the results. Comparison of the errors in the realised settling time with the theoretical ones in Table 3.1 shows some differences that are attributed to plant modelling errors but these are within acceptable limits for most applications.

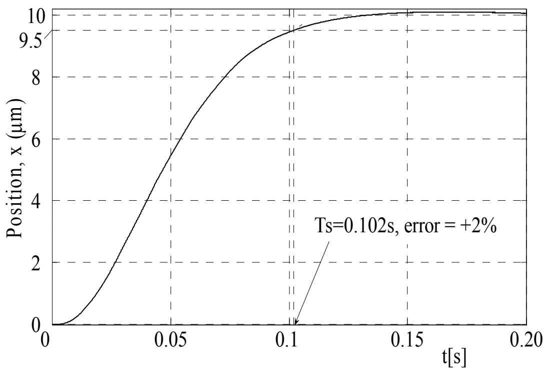


Fig. 4.3: Experimental IPD response (5% criterion).

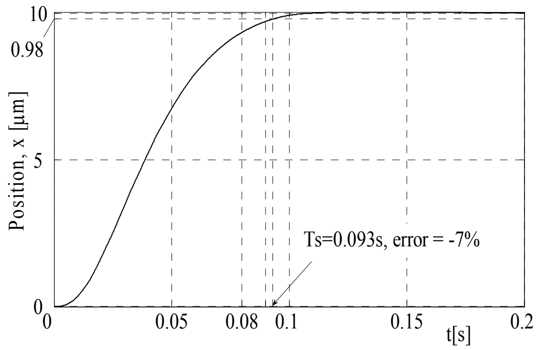


Fig. 4.4: Experimental IPD response (2% criterion).

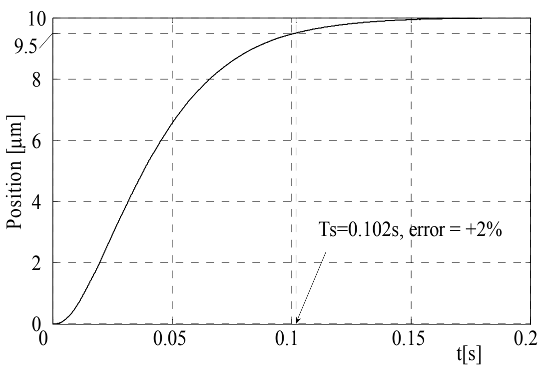


Fig. 4.5: Experimental SMC response (5% criterion).

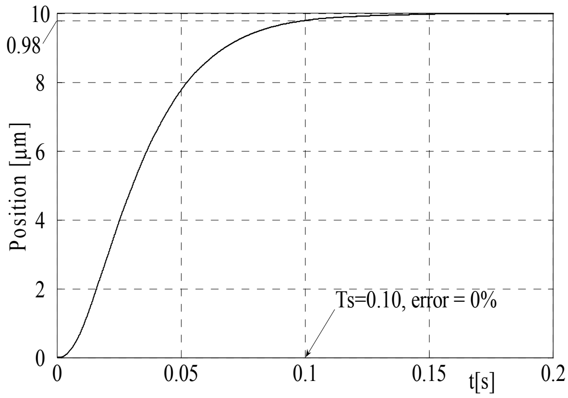


Fig. 4.6: Experimental SMC response (2% criterion).

5. Conclusions and Recommendations

Two settling time formulae have been derived for closed loop systems with coincident poles and their use in control systems design demonstrated. The experimental results show that the desired settling time is accurately realised. Extension to complex conjugate pole placement would be of interest as in a few cases a small amount of overshoot is desirable.

6. Acknowledgement

The author is very grateful to Paul Stadler of the Institute for Mechatronics at the Bern University of Science, Switzerland, for producing the DSpace based experimental results.

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